Topology and its Applications 35 (1990) 109-120
North-Holland

AN EXAMPLE IN THE DIMENSION THEORY OF METRIZABLE SPACES

John KULESZA
Department of Mathematical Sciences, George Mason University, Fairfax, VA 23185, USA

Received 14 July 1987
Revised 21 October 1988

A dense G_δ (completely) metrizable subspace Z of ${}^\omega\omega_1$ is described which satisfies $\text{Ind}(Z)=1$ while (trivially) $\text{ind}(Z)=0$. Necessarily, Z has weight equal to $\aleph_1$. Furthermore, it is shown that if C is a separable closed subspace of Z, and K is any other closed subspace of Z disjoint from C, then there is a clopen set O with $C\subset O\subset Z\setminus K$.

AMS (MOS) Subj. Class.: 54F45, 54G99

dimension full set metrizable space

Introduction

The first example of a gap metric space, a metrizable space for which the inductive dimensions ind and Ind do not coincide, Roy's space Δ, was described in 1962 (see [7, 8]). Since that time very few examples of such spaces have been forthcoming. A prominent example is Mrowka's space μ_0 (in [3]) which is N-compact. (Δ is not N-compact, as shown by Nyikos [5].) Another is a very recent example, assuming ♣, of Ostaszewski's [6], which has weight equal to $\aleph_1$. Ostaszewski's example gave an answer to a question, attributed to Kunen and van Douwen, of whether, in the absence of CH it was possible to have a gap-metric space of weight $\aleph_1$. This is because ♣ is independent of CH. (The author has recently been informed that Ostaszewski's construction does not need ♣, and can be carried out in ZFC.) Since ind and Ind coincide for separable metric spaces, $\aleph_1$ is as small as possible; assuming CH, the weight of Δ is $\aleph_1$, as the weight of Δ is known to be c.

In this paper, we describe a subspace Z of B, the Tychonov product of ω many copies of ω_1 (ω_1 with the order topology), which is a gap-metric space satisfying $\text{ind}(Z)=0$ and $\text{Ind}(Z)=1$. The space is in many ways more natural than existing examples, much easier to describe, and many of its important properties are fairly easy to verify. As a subspace of B, the weight of Z is $\aleph_1$ without making any special set-theoretic assumptions. Z_1 is described in Section 2, shown metrizable in Section 3, and $\text{Ind}(Z)$ is computed in Section 4.

In Section 5 we show that Z has the following property. If C is a closed separable subset of Z, and K is any other closed subset of Z not intersecting C, then there is a separation of Z between C and K; this is the content of Theorem 5.14. In this sense Z is more zero-dimensional than just having a base of clopen sets; all closed separable sets also have a "local base" of clopen sets. If C' and D' are closed sets which illustrate that $\operatorname{Ind}(Z)>0$, then the weight of each of C' and D' must be $\aleph_1$. This contrasts with all other gap-metric examples, which actually have pairs of separable sets which cannot be separated. In Section 6 we show that Z is complete, and indicate a proof that Z is not N-compact.

Our example is rooted in Roy's construction and, where possible, we point out the connections. Certain uses of symbols are to illuminate these connections; for example point sets P_1 and P_2^k correspond to similar point sets in Δ.

1. Preliminaries

For our purposes, a sequence will denote a function from ω or an element of ω into ω_1. If σ is a sequence, and D is a collection of sequences, then by $|\sigma|$ we mean domain(σ), and by $|D|$ we mean $\sup\{|\tau|: \tau\in D\}$. $\sigma|_n$, and $D|_n$ denote the appropriate restrictions to n ($n=\{0,1,2,\ldots,n-1\}$). If $(a_0,a_1,\ldots,a_{k-1})$ is an ordered k-tuple of elements of ω_1, we will consider it to be the same as the sequence ξ where $|\xi|=k$, and $\xi(i)=a_i$ for $i<k$. B $\sigma|_n^m$ where $m<n$ and $|\sigma|\geqslant n$ we mean the sequence $(\sigma(m),\ldots,\sigma(n-1))$. If $Y\subset\omega_1$ and $n\leqslant\omega$, by ${}^nY$ we mean $\{\sigma: |\sigma|=n$ and $\operatorname{image}(\sigma)\subset Y\}$.

Let ω_1 be equipped with the order topology; we partition ω_1 into L, the set of limit ordinals, and S, the set of nonlimit ordinals. We further partition S into sets $\{S_i: i\geqslant 1\}$ by letting $S_i=\{a+i: a\in L\}$; put the elements of ω in S_1. We let B denote the topological space which, as a point set is ${}^\omega\omega_1$, i.e., $B=\{\sigma: \sigma$ is a sequence and $|\sigma|=\omega\}$. We topologize B by giving it the product topology on ω many copies of ω_1. Our example Z is a dense G_δ subspace of B.

Concatenated sequences and groups of sequences are crucial for our purposes; we adopt the following notations. For sequences σ, τ and collections D, E of sequences with $|\sigma|<\omega$, and $|D|<\omega$:

$$\langle\sigma;\tau\rangle \text{ denotes the concatenation of } \sigma \text{ and } \tau,$$

that is

$$\langle\sigma;\tau\rangle=(\sigma(0),\sigma(1),\ldots,\sigma(|\sigma|),\tau(0),\tau(1),\ldots,\tau(|\tau|)),$$

$$\langle\sigma; E\rangle=\{\langle\sigma;\xi\rangle: \xi\in E\},$$

$$\langle D; E\rangle=\{\langle\delta;\xi\rangle: \delta\in D, \xi\in E\},$$

$$\overline{\langle\sigma; D\rangle}=\{\beta\in B: \text{there is an } n\in\omega \text{ such that } \beta|_n\in\langle\sigma; D\rangle\}.$$

(Later, when we are discussing the space Z, $\overline{\langle\sigma; D\rangle}$ will ambiguously mean $\overline{\langle\sigma; D\rangle}\cap Z$.)

2. The example Z

Z will denote the following subspace of B. $Z=\{\sigma\in B: \sigma(0)\in S,\ \sigma(k)\in L$ for at most one $k\in\omega\backslash\{0\}$, and if $\sigma(k)\in L$, then $\sigma(k+1)=\sigma(k)+k$ and $\sigma(k+i)\in S_k$ for $i\geq 2\}$.

It is more convenient to write $Z=P_1\cup P_2^1\cup P_2^2\cup P_2^3\cup\cdots$ where

$$P_1=\{\sigma\in B: \sigma(i)\in S \text{ for all } i\in\omega\},$$

and for each $k\in\omega\backslash\{0\}$

$$P_2^k=\bigcup_{a\in L}\{\sigma\in B: \sigma(i)\in S \text{ for all } i<k,\ \sigma(k)=a,\ \sigma(k+1)=a+k, \text{ and for all } j\geq 2,\ \sigma(k+j)\in S_k\}.$$

The reader familiar with Δ will note the similarities between the points P_1 and P_2 in Δ, and those here.

We will construct a standard base for Z. First we fix a standard base at points of L in ω_1. For each $a\in L$, let $\tilde{a}$ be an increasing sequence in ω_1 whose limit is a, and let $M_m(a)=(\tilde{a}(m),a]$. Clearly $\{M_m(a): m\in\omega\}$ is a nested local base at a in ω_1.

Now,

(i) for $\sigma\in P_1$, $m\in\omega$, let

$$N_m(\sigma)=\{\tau\in Z: \tau|_{m+1}=\sigma|_{m+1}\},$$

(ii) for $\xi\in P_2^k$, where $k\geq 1$, $m>k$ let

$$N_m(\xi)=\{\tau\in Z: \tau|_k=\xi|_k,\ \tau(k)\in M_m(\xi(k)) \text{ and } \tau|_{m+1}^{k+1}=\xi|_{m+1}^{k+1}\}.$$

For $m\leq k$ appropriately delete the restrictions on coordinates $>m$.

It is easy to check that the sets

$$\{N_m(\sigma): m\in\omega,\ \sigma\in P_1\}\cup\bigcup_{k\geq 1}\{N_m(\xi): \xi\in P_2^k,\ m\in\omega\}$$

form a basis for the topology Z inherits as a subspace of B, and as a subspace of B, it is obvious that Z is zero-dimensional ($\operatorname{ind}(Z)=0$), and that $\operatorname{weight}(Z)\leq\aleph_1$. When we show that Z is metrizable and $\operatorname{Ind}(Z)>0$, it will follow that $\operatorname{weight}(Z)=\aleph_1$ because the inductive dimensions agree for separable metric spaces.

Observe that, for $\sigma\in P_1$, $N_m(\sigma)$ is also a basic neighborhood of σ in Z if we use the discrete topology on ω_1 instead of the order topology. For $\sigma\in P_2$ it is almost such a basic set, the "almost" because of the one coordinate in which σ takes on a limit ordinal value; it is a countable union of such basic sets.

3. Z is metrizable

We will make use of the following theorem, due to Frink; see [4, p. 33].

3.1. Theorem. *A Hausdorff space X is metrizable if and only if there is a countable open neighborhood base $\{V_i(p): i\geq 1\}$ for each $p\in X$ satisfying:*

For each $p \in X$, and each positive integer i, there is a positive integer t such that $V_t(p) \cap V_t(q) \neq \emptyset$ implies $V_t(q) \subset V_i(p)$.

We will also make use of the next four lemmas, which illustrate the relationships among various types of points in Z.

3.2. Lemma. *If σ, $\tau \in P_2^k$, and $m > k$, then $N_m(\sigma) \cap N_m(\tau) = \emptyset$ or else $N_m(\sigma) = N_m(\tau)$.*

Proof. $\sigma(k+1) = \sigma(k) + k$, and $\tau(k+1) = \tau(k) + k$. From the definition of the N_m, $N_m(\sigma) \cap N_m(\tau) \neq \emptyset$ implies that $\sigma(i) = \tau(i)$ for $i \leqslant m$ and $i \neq k$. But then $\sigma(k+1) = \tau(k+1)$, and from the first statement, we conclude that $\sigma(k) = \tau(k)$, and so $M_m(\sigma(k)) = M_m(\tau(k))$ and this is enough to guarantee that $N_m(\sigma) = N_m(\tau)$. $\square$

3.3. Lemma. *If $m > k_1 > k_2 \geqslant 1$ and $\sigma_1 \in P_2^{k_1}$, $\sigma_2 \in P_2^{k_2}$ then $N_m(\sigma_1) \cap N_m(\sigma_2) = \emptyset$.*

Proof. Suppose $\tau \in N_m(\sigma_1)$, then clearly $\tau(m) = \sigma_1(m)$, and so $\tau(m) \in S_{k_1}$. Thus since $S_{k_1} \cap S_{k_2} = \emptyset$, $\tau(m) \notin S_{k_2}$, but $\sigma_2(m) \in S_{k_2}$, and it follows that $\tau \notin N_m(\sigma_2)$. $\square$

3.4. Lemma. *If $\sigma \in P_1$, $m \geqslant 1$, then there is a t such that $N_t(\sigma) \cap N_t(\tau) = \emptyset$ for all $\tau \in P_2^1 \cup P_2^2 \cup \cdots \cup P_2^m$.*

Proof. $\sigma(m+1)$ is in at most one element of $\{S_1, S_2, \ldots, S_m\}$; if it is not in any, then let $t = m+1$, and the result follows. So suppose $\sigma(m+1) \in S_k$ where $k \leqslant m$. Already $N_{m+1}(\sigma) \cap N_{m+1}(\tau) = \emptyset$ for $\tau \in P_2^i$, $i \leqslant m$ and $i \neq k$, since $\sigma(m+1) \neq \tau(m+1)$ for such a τ. If $\sigma(k+1) \notin S_k$, then let $t = m+1$; otherwise $N_{m+1}(\sigma) \cap N_{m+1}(\xi) \neq \emptyset$ where $\xi \in P_2^k$ implies that $\xi(k) = \sigma(k+1) - k \in L$. Now $\sigma(k) \in S$, so $\sigma(k) \neq \xi(k)$, and there is an r so that $M_r(\xi(k))$ does not contain $\sigma(k)$, hence $N_r(\xi) \cap N_r(\sigma) = \emptyset$, and we may choose $t = \max\{m+1, r\}$. $\square$

3.5. Lemma. *Suppose $\sigma \in Z$, $m \geqslant 1$, and $\tau \notin P_2^1 \cup \cdots \cup P_2^m$. Then if $t \geqslant m$, $N_t(\tau) \cap N_m(\sigma) = \emptyset$ or $N_t(\tau) \subset N_m(\sigma)$.*

Proof. $N_t(\tau) \subset N_m(\tau)$, and from the definition of $N_m(\tau)$ and the fact that $\tau \notin P_2^1 \cup \cdots \cup P_2^m$, $N_m(\tau) = \{\xi \in Z: \xi|_{m+1} = \tau|_{m+1}\}$. It is easy to check that if $\delta \in N_m(\sigma)$, then $\langle \delta|_{m+1}; \emptyset \rangle$ is a subset of $N_m(\sigma)$. Hence $N_m(\tau) \cap N_m(\sigma) = \emptyset$ or else $N_m(\tau) \subset N_m(\sigma)$. $\square$

We now show, using the collections $\{N_m(\xi): m \geqslant 1\}$ for each $\xi \in Z$, that the Frink condition is met. Suppose $\sigma \in Z$, and $i \geqslant 1$ are given. We need to produce $t \geqslant 1$ such that $N_t(\tau) \cap N_t(\sigma) \neq \emptyset$ implies $N_t(\tau) \subset N_m(\sigma)$. We consider separately the cases where $\sigma \in P_1$ and where $\sigma \notin P_1$.

Case 1: $\sigma \in P_1$. By Lemma 3.4, there is a t_1 such that if $\tau \in P_2^1 \cup \cdots \cup P_2^i$, then $N_{t_1}(\tau) \cap N_{t_1}(\sigma) = \emptyset$. Let $t = \max\{t_1, i\}$. If $N_t(\tau) \cap N_t(\sigma) \neq \emptyset$, then $\tau \notin P_2^1 \cup \cdots \cup P_2^i$, but then by Lemma 3.5, $N_t(\tau) \subset N_i(\sigma)$.

Case 2: $\sigma \notin P_1$. Assume $\sigma \in P_2^k$. We may also assume $i > k$. Let $t = i + 1$. By Lemma 3.3, for $m \leq i$, and $m \neq k$, and $\tau \in P_2^m$, $N_t(\tau) \cap N_t(\sigma) = \emptyset$. For $\tau \in P_2^k$, by Lemma 3.2, $N_t(\tau) = N_t(\sigma)$ or $N_t(\tau) \cap N_t(\sigma) = \emptyset$. Otherwise, by Lemma 3.5, if $N_t(\tau) \cap N_t(\sigma) \neq \emptyset$, then $N_t(\tau) \subset N_i(\sigma)$. □

4. $\mathrm{Ind}(Z) = 1$

First we show $\mathrm{Ind}(Z) \leq 1$. In view of Lemmas 3.3 and 3.4, it can be seen that each of the sets P_2^k is closed. From Lemma 3.2, P_2^k has a sequence of covers which are order 1 at each level whose union forms a base for P_2^k; by [4, Theorem 12.6], $\mathrm{Ind}(P_2^k) = 0$. Thus by [4, Theorem 10.4], $\mathrm{Ind}(\bigcup_{k>0} P_2^k) = 0$. It is clear that $\mathrm{Ind}(P_1) = 0$, as P_1 is a countably infinite product of discrete spaces. Z is thus a metrizable union of two spaces with $\mathrm{Ind} = 0$, and so by [4, Theorem 12.6], $\mathrm{Ind}(Z) \leq 1$.

We now show $\mathrm{Ind}(Z) \geq 1$. Roy's proof that $\mathrm{Ind}(\Delta) > 0$ has two critical components. The first is the "grabbing" at individual levels; this is realized by the use of all injective sequences of positive reals, and insures that, given an appropriate pair C_1, C_2 of closed sets, there are points of Δ fairly close to both C_1 and C_2. The second component makes sure there are enough of these points to force a similar situation at a lower level. To this end he invented indicators.

Our proof that $\mathrm{Ind}(Z) > 0$ is largely based on Roy's proof for Δ; the following should help clarify the differences.

The "grabbing" in Z cannot possibly hope to have the equivalent of all sequences at its disposal. We instead make use of the following special case of the more general Pressing Down Lemma. For a proof see [2, Lemma 6.15, p. 80].

4.1. Theorem. *Suppose $f: L \to \omega_1$ is such that $f(a) < a$ for each $a \in L$. Then there is a $b \in \omega_1$ such that* $\mathrm{card}(f^{-1}(b)) = \aleph_1$.

In order to make our application of Theorem 4.1 more clear, we prove now a lemma which illustrates our use of it.

4.2. Lemma. *For each $a \in L$, let n_a be a positive integer, and let $M = \{M_{n_a}(a): a \in L\}$. Then there is a $b \in \omega_1$ such that for all $c > b$, c is in $\aleph_1$ many elements of M. Furthermore, if $M = \bigcup_{i \in \omega} M_i$ where $\{M_i: i \in \omega\}$ is any collection of subsets of M, there is a $j \in \omega$ such that for all $c > b$, c is in $\aleph_1$ many elements of M_j.*

Proof. Define $f: L \to \omega_1$ by $f(a) = \tilde{a}(n_a)$. f satisfies the hypothesis of Theorem 4.1, hence we can choose b so that $\mathrm{card}(f^{-1}(b)) = \aleph_1$. If $c \in \omega_1$, and $c > b$, then $\mathrm{card}(\{d > c: d \in f^{-1}(b)\}) = \aleph_1$, and for any d in that set, $c \in M_{n_d}(d)$. The second part follows because for some j, $M_j \cap f^{-1}(b)$ is uncountable, and thus cofinal in ω_1. □

One should think of the neighborhoods of points of L as being the Z equivalents of the y-sequences in the definition of Δ. Observe that there are c many y-sequences, while there are only $\aleph_1$ many neighborhoods of points in L. This is the main reason that the weight of Z can be $\aleph_1$.

To do the work of Roy's indicators we use the notion of a full set. This has been used by Fleissner for a different purpose. The following definition and two lemmas are essentially found in [1].

4.3. Definition. A set $T \subseteq {}^n\omega_1$ is *full* iff for all $\sigma \in T$, and all $j < n$, $\{\tau(j): \sigma|_j = \tau|_j$ and $\tau \in T\}$ is uncountable.

Thus T is full when the tree of initial subsequences of members of T branches uncountably at each nonterminal mode.

The principal combinatorial properties of full sets are given in the following lemmas.

4.4. Lemma. *If $T \subseteq {}^n\omega_1$ is full, and $g: T \to \omega$ is a function, then there is an $m \in \omega$ such that $g^{-1}(m)$ contains a full subset of T.*

4.5. Lemma. *If R is an uncountable set (subset of ω_1), and $\{C_m: m \in \omega\}$ is such that $C_m \subseteq {}^mR$ for each m, and for each $\sigma \in {}^\omega R$, there is an n such that $\sigma|_n \in C_n$, then there is a t such that C_t contains a full subset of ${}^tR$.*

We now prove important lemmas which allow us to conclude $\mathrm{Ind}(Z) \geq 1$.

4.6. Lemma. *If $(a_0, a_1, \ldots, a_{k-1})$ is an ordered k-tuple of elements of S, $a_k \in L$, and $\{U_1, U_2\}$ is an open cover of Z, then there is an $n \in \omega$, an $i \in \{1, 2\}$ and a full subset D of ${}^nS_k$ such that*

$$\sigma \in \overline{\langle (a_0, a_1, \ldots, a_{k-1}, a_k, a_k + k); D\rangle}$$

implies $N_{k+n+1}(\sigma) \subset U_i$.

Proof. For each $\xi \in {}^\omega S_k$, let σ_ξ denote $\langle (a_0, a_1, \ldots, a_{k-1}, a_k, a_k + k); \xi\rangle \in Z$.

For $m \in \omega$, let $C_m = \{\xi|_m: N_{k+m+1}(\sigma_\xi) \subset U_1$ or $U_2\}$. The conditions of Lemma 4.5 are met (R replaced by S_k), so there is an $n \in \omega$ such that C_n contains a full subset of ${}^nS_k$, call this set D'.

If $d \in D'$, and $\sigma_1, \sigma_2 \in \overline{\langle (a_0, \ldots, a_k, a_k + k); d\rangle}$, then $N_{k+n+1}(\sigma_1) = N_{k+n+1}(\sigma_2)$ and there is i_d with $N_{k+n+1}(\sigma_1) \subset U_{i_d}$. By Lemma 4.4, one of $\{d: i_d = 1\}$ and $\{d: i_d = 2\}$ contains a full subset D of D'. $\square$

4.7. Lemma. *If $(a_0, \ldots, a_{k-1})$ is an ordered k-tuple in S, $\{U_1, U_2\}$ an open cover of Z, n_1, $n_2 \in \omega$, and D_1, D_2 are full subsets of ${}^{n_1}S$, ${}^{n_2}S$ respectively satisfying $\overline{\langle (a_0, \ldots, a_{k-1}; D_i\rangle} \subset U_i$ for $i \in \{1, 2\}$, then there are $a_k \in S$, n_1', $n_2' \in \omega$, D_1', D_2' full subsets of ${}^{n_1'}S$, ${}^{n_2'}S$ respectively satisfying $\overline{\langle (a_0, \ldots, a_{k-1}, a_k); D_i'\rangle} \subset U_i$ for $i \in \{1, 2\}$.*

Proof. For each $a \in L$, apply Lemma 4.6 to get n_a, i_a and a full subset D_a of ${}^{n_a}S$ such that $N_{k+n_a+1}(\sigma) \subset U_{i_a}$ whenever $\sigma \in \overline{\langle (a_0, \ldots, a_{k-1}, a, a+k); D_a \rangle}$. By Lemma 4.2, there is $b \in \omega_1$ such that for all $c > b$, c is in $\aleph_1$ many members of $M = \{M_{k+n_a+1}(a): a \in L\}$. For $r \in \{1, 2\}$ and $s \in \omega$ let $L_{r,s} = \{a \in L: i_a = r$, and $n_a = s\}$, and let $M_{r,s} = \{M_{k+s+1}(a): a \in L_{r,s}\}$. $\{M_{r,s}\}$ is a countable partition of M, so there is a pair r', s' hereafter called r, s for simplicity such that if $c > b$, then c is in $\aleph_1$ many members of $M_{r,s}$, by the second part of Lemma 4.2. Now suppose w.l.o.g. that $r = 1$. Since $D_2|_1$ is uncountable (because D_2 is full), there is a $\sigma \in D_2|_1$ with $\sigma(0) > b$. Let $a_k = \sigma(0)$, $n_2' = n_2 - 1$ and define $D_2' = \{\tau|_{n_2}^1: \tau \in D_2$ and $\tau(0) = \sigma(0)\}$ if $n_2 > 0$; otherwise let $n_2' = 0$ and choose D_2' to be any uncountable subset of S. In either event

$$\overline{\langle (a_0, \ldots, a_k); D_2' \rangle} \subset \overline{\langle (a_0, \ldots, a_{k-1}); D_2 \rangle} \subset U_2.$$

Now let $n_1' = s+1$, and $D_1' = \{\sigma$: there is $a \in L_{r,s}$ such that $\sigma(0) = a + k$, and $\sigma|_{n_1'}^1 \in D_a\}$. If $\xi \in \overline{\langle (a_0, \ldots, a_k); D_1' \rangle}$, then $\xi(i) = a_i$ for $i < k$, $\xi_1(k) = a_k$ and there is $a \in L_{r,s}$ so that $\xi(k+1) = a + k$, and $\xi|_{k+2+s}^{k+2} \in D_a$. Furthermore $\xi(k) \in M_{k+s+1}(a)$. Thus $\xi \in N_{k+s+1}(\sigma)$ where $\sigma \in \overline{\langle (a_0, \ldots, a_{k-1}, a, a+k); D_a \rangle}$ and hence $\xi \in U_1$. □

4.8. Lemma. *If $(a_0, a_1, \ldots, a_{k-1})$ is an ordered k-tuple of elements of S, $\{U_1, U_2\}$ is an open cover of Z, n_1, $n_2 \in \omega$, and D_1, D_2 are full subsets of ${}^{n_1}S$ and ${}^{n_2}S$ respectively such that $\overline{\langle (a_0, \ldots, a_{k-1}); D_i \rangle} \subset U_i$ for $i \in \{1, 2\}$, then $U_1 \cap U_2 \neq \emptyset$.*

Proof. The hypotheses of this lemma allow us to apply Lemma 4.7 and get $a_k \in S$, $n_1^{(1)}$, $n_2^{(1)}$, $D_1^{(1)}$ and $D_2^{(1)}$ full subsets of ${}^{n_1^{(1)}}S$ and ${}^{n_2^{(1)}}S$ respectively such that $\overline{\langle (a_0, \ldots, a_{k-1}, a_k); D_i^{(1)} \rangle} \subset U_i$, and we have essentially reproduced our hypotheses one level down. Thus we can successively apply Lemma 4.7 to get a_{k+1}, a_{k+2}, and full sets $D_1^{(2)}$, $D_2^{(2)}$ etc. such that $\overline{\langle (a_0, \ldots, a_{k+n}); D_i^{(n+1)} \rangle} \subset U_i$. Now let $\zeta = (a_0, a_1, a_2, \ldots) \in P_1$. From the previous statement $N_n(\zeta) \cap U_i \neq \emptyset$, for each $i \in \{1, 2\}$. Thus $\zeta \in \mathrm{cl}(U_1) \cap \mathrm{cl}(U_2)$, and since $U_1 \cup U_2 = Z$, it follows that $U_1 \cap U_2 \neq \emptyset$. □

We will now show that $\mathrm{Ind}(Z) > 0$. For $i \in \{1, 2\}$ let $D_i = \{\sigma \in {}^2S: \sigma(0) \in S_{2i}$ and $\sigma(1) \in S_{2i+1}\}$, and let $C_i = \mathrm{cl}(\overline{\langle (0); D_i \rangle})$. We will show that $C_1 \cap C_2 = \emptyset$. From Lemma 4.8 it is then obvious that there is no separation of Z between C_1 and C_2, hence $\mathrm{Ind}(Z) \geq 1$.

First observe that no point of P_2^1 is in $C_1 \cup C_2$ because if $\sigma \in P_2^1$, then $\sigma(2) \in S_1$ and so $N_2(\sigma)$ misses $\overline{\langle (0); D_i \rangle}$ for each i. If $\sigma \notin P_2^1$, then $\sigma(1)$ is in at most one element of $\{S_2, S_4\}$ and so $N_1(\sigma)$ misses at least one of $\overline{\langle (0); D_i \rangle}$. Hence $C_1 \cap C_2 = \emptyset$.

5.

We show that if C is a closed separable subspace of Z, and K is any other closed subspace of Z which does not intersect C, then there is a separation of Z between C and K, i.e., there is a clopen set O with $C \subset O \subset Z \setminus K$.

Although Z is metrizable it does not come with a metric, thus many of the properties of Z are phrased in terms of a basis. The following notation, definition and lemmas should be viewed with that in mind. For a collection G of subsets of a set X, and $A \subset X$ by $\mathrm{st}_G(A)$ $(=\mathrm{st}_G^1(A))$ we mean $\bigcup\{g \in G: A \cap g \neq \emptyset\}$; inductively, for $n > 1$ by $\mathrm{st}_G^n(A)$ we mean $\mathrm{st}_G(\mathrm{st}_G^{n-1}(A))$.

5.1. Definition. A collection $\{G_n: n \in \omega\}$ of open covers of a space X is a *strong development* for X if G_{i+1} always refines G_i and for each $x \in X$ the set $\{\mathrm{st}_{G_n}^2(\{x\}): n \in \omega\}$ is a local base at x.

We will need the following two lemmas on strong developments.

5.2. Lemma. *If $\{G_i: i \in \omega\}$ is a strong development for X, $p \in X$ and O is open with $p \in O$, then given $n \in \omega$, there is $m \in \omega$ such that $\mathrm{st}_{G_m}^n(\{p\}) \subset O$.*

Proof. The case where $n = 2$ is given by the definition of strong development. Assuming there is $m(n-1)$ such that $\mathrm{st}_{G_{m(n-1)}}^{n-1}(\{p\}) \subset O$, choose $m(n) > m(n-1)$ so that $\mathrm{st}_{G_{m(n)}}^2 \subset \mathrm{st}_{G_{m(n-1)}}(\{p\})$, using the fact that $\mathrm{st}_{G_{m(n-1)}}(\{p\})$ is open. The lemma follows. □

5.3. Lemma. *If $G = \{G_i: i \in \omega\}$ is a strong development for X, C is a closed subset of X, $n \geq 1$, and $O = \bigcup\{O_i: i \in \omega\}$ where for each i, O_i is open and $O_i \subset \mathrm{st}_{G_i}^n(C)$, then $\mathrm{Bd}(O)\backslash(\bigcup_{i\in\omega} \mathrm{Bd}(O_i)) \subset C$.*

Proof. Suppose $p \in \mathrm{Bd}(O)\backslash(\bigcup_{i\in\omega} \mathrm{Bd}(O_i))$. For each $i \in \omega$, choose g_i so that $p \in g_i \in G_i$. There is always $m_i > i$ such that $g_i \cap O_{m_i} \neq \emptyset$. Now $\mathrm{st}_{G_{m_i}}^n(C) \subset \mathrm{st}_{G_i}^n(C)$, and so $p \in \mathrm{st}_{G_i}^{n+1}(C)$, hence $\mathrm{st}_{G_i}^{n+1}(\{p\}) \cap C \neq \emptyset$ for all i. Using Lemma 5.2 we conclude $p \in C$. □

We now examine Z. Letting $G_0 = Z$, and for $i > 0$, $G_i = \{N_i(\sigma): \sigma \in Z\}$ we have that $G = \{G_i: i \in \omega\}$ is a strong development for Z. By saying $A \subset P_2^k$ is *initially separable* we mean $\{a: \sigma(i) = a$ for some $\sigma \in A$, $i \leq k\}$ is a countable set.

Our goal is Theorem 5.14. Initially separable sets are essential to achieving this goal. The key facts about these sets are the content of Lemmas 5.6 and 5.7. We then use these facts to prove special cases of Theorem 5.14, namely Lemma 5.10 and Lemma 5.13, the proof of Theorem 5.14 shows how to combine these two lemmas in order to get the main result.

5.4. Lemma. *If $n, k \geq 1$, $\sigma \in P_2^k$, $\{a_0, \ldots, a_n\} \subset S$, then letting $\hat{N}_k(\sigma) = N_k(\sigma)\backslash\{\tau: \tau(k) \in \{a_0, \ldots, a_n\}\}$, $\hat{N}_k(\sigma)$ is a clopen subset of $N_k(\sigma)$ containing $\{\xi \in Z: \xi|_{k+1} = \sigma|_{k+1}\}$.*

Proof. Let $H_i = \{\tau: \tau|_k = \sigma|_k$, and $\tau(k) = a_i\}$. Then H_i is clopen, and so $\bigcup_{i=0}^{n-1} H_i$ is clopen. Thus $\hat{N}_k(\sigma) = N_k(\sigma)\backslash(\bigcup_{i=0}^{n-1} H_i)$ is clopen (as $N_k(\sigma)$ is clopen). No point of $\{\xi: \xi|_{k+1} = \sigma|_{k+1}\}$ is in any H_i because $\sigma(k) \in L$, and $\{a_0, \ldots, a_n\} \subset S$. □

5.5. Lemma. *Let $\{\sigma_i: i\in\omega\}$ be a collection of elements of P_2^k. There is a clopen set O contained in $\bigcup\{N_{k-1}(\sigma_i): i\in\omega\}$ which contains $\{\alpha: \alpha|_{k+1}=\sigma_i|_{k+1}$ for some $i\in\omega\}$.*

Proof. Let $a=\operatorname{lub}\{\sigma_i(k): i\in\omega\}$, let $\{b_j: j\in\omega\}$ be an enumeration of $L\cap[0,a]$ and let $\{c_j: j\in\omega\}$ be an enumeration of $S\cap[0,a]$. For each pair i, j, choose $\xi_{i,j}\in\overline{\langle\sigma_i|_k; (b_j)\rangle}$, and let $T=\{\tau_i: i\in\omega\}$ be a listing of these. Let $O'=\bigcup\{N_k(\tau_i): i\in\omega\}$. O' is contained in $\bigcup\{N_{k-1}(\sigma_i): i\in\omega\}$, and contains $\{\alpha: \alpha|_{k+1}=\sigma_i|_{k+1}$ for some $i\in\omega\}$. Furthermore O' has no boundary in $P_1\cup P_2^k\cup P_2^{k+1}\cup\cdots$. For elements of $P_1\cup P_2^{k+1}\cup P_2^{k+2}\cup\cdots$ this is obvious from Lemma 3.5. For $\alpha\in P_2^k$, $\alpha\notin O'$, either $\alpha|_k\notin T|_k$ in which case $N_{k-1}(\alpha)\cap O'=\emptyset$ or else $\alpha(k)>a$. In this case there is an $m>k$ such that $M_m(\alpha(k))\cap[0,a]=\emptyset$, and thus $N_m(\alpha)\cap O'=\emptyset$.

We now shrink O' enough to remove the boundary in $P_2^1\cup\cdots\cup P_2^{k-1}$; we are careful not to add any other boundary.

For each $\tau_i\in T$, let $\hat{N}_k(\tau_i)=N_k(\tau_i)\backslash\{\alpha: \alpha(k)\in\{c_0,\ldots,c_i\}\}$. Let $O=\bigcup\{\hat{N}_k(\tau_i): i\in\omega\}$. Then, by Lemma 5.4, O is open and $O\cap P_2^k=O'\cap P_2^k$, so since $O\subset O'$, O has no boundary in P_2^k. Clearly we have added no boundary in $P_1\cup P_2^{k+1}\cup P_2^{k+2}\cup\cdots$. Also $O\subset\{N_{k-1}(\sigma_i): i\in\omega\}$.

We need only to check that O has no boundary in $P_2^1\cup\cdots\cup P_2^{k-1}$. Suppose $\xi\in P_2^m$ where $m<k$. Consider $\xi(k)$. If $\xi(k)\notin S\cap[0,a]$, then $N_k(\xi)\cap O=\emptyset$. Otherwise suppose $\xi(k)=c_{i_\xi}$. Then $N_k(\xi)\cap\hat{N}_k(\tau_i)\neq\emptyset$ implies $i\leq i_\xi$. But then $\{\tau_j(m): j\leq i_\xi\}$ is a finite subset of S missing $\xi(m)$, and there is $n>k$ so that $M_n(\xi(m))\cap\{\tau_j(m): j\leq i_\xi\}=\emptyset$, and so $N_n(\xi)\cap O=\emptyset$. □

5.6. Lemma. *Let $T\subset P_2^k$ be initially separable. There is a clopen set O_T such that $T\subset O_T\subset \operatorname{st}_{G_{k-1}}(T)$.*

Proof. Since T is initially separable, $T|_{k+1}$ is countable. Let $\{\sigma_i: i\in\omega\}$ be a subset of P_2^k such that $T|_{k+1}=\{\sigma_i|_{k+1}: i\in\omega\}$. Then apply Lemma 5.5. □

5.7. Lemma. *If $\{N_{n_i}(\sigma_i): i\in\omega\}$ is a collection of elements of G, then for each $k\geq 1$, $\operatorname{Bd}(\bigcup\{N_{n_i}(\sigma_i): i\in\omega\})\cap P_2^k$ is an initially separable subset of P_2^k.*

Proof. Let $a=\operatorname{lub}\{\sigma_i(j): i\in\omega, j\leq k\}$, and let $I_k=\{\sigma\in P_2^k: \sigma(i)\leq a$ for $i\leq k\}$. I_k is initially separable since a is a countable ordinal. If $\tau\in P_2^k\backslash I_k$, then for some $i\leq k$, $\tau(i)>a$. Thus we can find an n_τ such that $\xi\in N_{n_\tau}(\tau)$ implies $\xi(i)>a$, and so $\tau\notin\operatorname{Bd}(\bigcup\{N_{n_i}(\sigma_i): i\in\omega\})$. Hence the boundary in P_2^k is a subset of I_k, hence is initially separable. □

5.8. Lemma. *If $C\subset P_2^k$ is separable, $n>k$, then setting $C^n=\bigcup\{N_n(\sigma): \sigma\in C\}$, $\operatorname{Bd}(C^n)$ is an initially separable subset of P_2^n.*

Proof. Since C is separable, $\{N_n(\sigma): \sigma\in C\}$ is countable, and thus Lemma 5.7 gives that $\operatorname{Bd}(C^n)\cap P_2^n$ is initially separable. It is routine to verify that C^n has no other boundary. □

5.9. Lemma. *If C is a closed separable subset of P_2^k, and $n > k$, there is a clopen set O_C such that $C \subset O_C \subset \mathrm{st}^2_{G_n}(C)$.*

Proof. Let $m = n+1$. By Lemma 5.8, setting $O^m = \bigcup\{N_m(\sigma)\colon \sigma \in C\}$, we have that $C \subset O^m$, and $\mathrm{Bd}(O^m)$ is an initially separable subset of P_2^m. Also $O^m \subset \mathrm{st}_{G_n}(C)$. By Lemma 5.6 there is a clopen set O with $\mathrm{Bd}(O^m) \subset O \subset \mathrm{st}_{G_{m-1}}(\mathrm{Bd}(O^m)) = \mathrm{st}_{G_n}(\mathrm{Bd}(O^m))$. Let $O_C = O^m \cup O$. □

5.10. Lemma. *If C is a closed separable subset of P_2^k, and K is another closed set with $C \cap K = \emptyset$, then there is a clopen set O with $C \subset O \subset Z \setminus K$.*

Proof. For each $i > k$, let $C_i = \{\sigma \in C\colon \mathrm{st}^2_{G_i}(\{\sigma\}) \cap K = \emptyset\}$. Then $C = C_{k+1} \cup C_{k+2} \cup \cdots$, and each C_i is closed. Apply Lemma 5.9 to each C_i to get a clopen O_{C_i} with $C_i \subset O_{C_i} \subset \mathrm{st}^2_{G_i}(C_i)$; thus $O_{C_i} \cap K = \emptyset$. Let $O = \bigcup_{i>k} O_{C_i}$. O is open, contains C and $O \subset Z \setminus K$. By Lemma 5.3, $\mathrm{Bd}(O) \subset C$, and hence is empty, because $C \subset O$. □

Lemmas 5.11–5.13 are for the case $C \subset P_1$.

5.11. Lemma. *If $C \subset P_1$ is closed and separable, and K is any other closed set not intersecting C, then there is a countable collection D of sets in G such that $C \subset \bigcup D \subset$ $\mathrm{cl}(\bigcup D) \subset Z \setminus K$, and $\mathrm{Bd}(\bigcup D) \cap P_1 = \emptyset$.*

Proof. There is an open set O with $C \subset O \subset \mathrm{cl}(O) \subset Z \setminus K$. For each $\sigma \in C$, let n_σ be such that $N_{n_\sigma}(\sigma) \subset O$. As C is separable $\{N_{n_\sigma}(\sigma)\colon \sigma \in C\}$ has a countable subcover, call this collection D. Using Lemma 5.3 it is clear that $\bigcup D$ has no boundary in P_1. □

5.12. Lemma. *If $C \subset P_1$ is closed and separable, $n \geq 1$, there is a clopen O_C with $C \subset O_C \subset \mathrm{st}^3_n(C)$.*

Proof. The set $V = P_2^1 \cup \cdots \cup P_2^n \cup (Z \setminus \mathrm{st}_{G_n}(C))$ is a closed set missing C. Apply Lemma 5.11 to get D_a countable subset of G with $C \subset \bigcup D \subset \mathrm{cl}(\bigcup D) \subset Z \setminus V$, and $\mathrm{Bd}(\bigcup D) \cap P_1 = \emptyset$. Thus $\bigcup D \subset \mathrm{st}_{G_n}(C)$, and $\mathrm{Bd}(\bigcup D) \subset P_2^{n+1} \cup P_2^{n+2} \cup \cdots$. Since D is countable, $\mathrm{Bd}(\bigcup D) \cap P_2^{n+i}$ is initially separable, by Lemma 5.7. Thus by Lemma 5.9, there is a clopen O_{n+i} with $\mathrm{Bd}(\bigcup D) \cap P_2^{n+i} \subset O_{n+i} \subset \mathrm{st}^2_{G_{n+i-1}}(\mathrm{Bd}(\bigcup D) \cap P_2^{n+i})$, thus $O_{n+i} \subset \mathrm{st}^3_{G_n}(C)$. Let $O_C = \bigcup_{i=1}^\infty O_{n+i} \cup (\bigcup D)$. Obviously $C \subset O_C \subset \mathrm{st}^3_{G_n}(C)$, and O_C is open. O_C is closed by Lemma 5.3, using $\mathrm{cl}(\bigcup D)$ as the closed set. □

5.13. Lemma. *If C is a closed separable subset of P_1, and K is any other closed set disjoint from C, then there is a clopen set O with $C \subset O \subset Z \setminus K$.*

Proof. The proof is now similar to the proof of Lemma 5.10. This time let $C_i = \{\sigma \in C\colon \mathrm{st}^3_{G_i}(\{\sigma\}) \cap K = \emptyset\}$, $C = C_1 \cup C_2 \cup \cdots$ and each C_i is closed. Now apply

Lemma 5.12 to C_i to get a clopen O_{C_i} with $C_i \subset O_{C_i} \subset \mathrm{st}^3_{G_i}(C)$, and let $O = \bigcup_{i=1}^{\infty} O_{C_i}$. $\square$

Finally we can prove:

5.14. Theorem. *If C is a separable closed subset of Z, and K is any other closed subset of Z not intersecting C, then there is a clopen set O with $C \subset O \subset Z \backslash K$.*

Proof. For $i \geq 1$, let $C_i = C \cap P_2^i$. C_i is closed. By Lemma 5.10 there is a clopen O_{C_i} with

$$C_i \subset O_{C_i} \subset Z \backslash (K \cup (Z \backslash \mathrm{st}_{G_i}(C_i))).$$

Let $O_f = \bigcup_{i=1}^{\infty} O_{C_i}$. Lemma 5.3 guarantees that $\mathrm{Bd}(O_f) \subset C$, hence $\mathrm{Bd}(O_f) \subset C \cap P_1$. But now $C_\infty = C / O_f$ is a closed separable subset of P_1, containing $\mathrm{Bd}(O_f)$. Apply Lemma 5.13 to get a clopen O_{C_∞} with $C_\infty \subset O_{C_\infty} \subset Z \backslash K$. The desired set is $O = O_f \cup O_{C_\infty}$. $\square$

6.

In Section 6.1 we show that Z is complete, and in Section 6.2 we indicate how to show Z is not N-compact.

6.1. Z is complete

Z is, in fact, a dense G_δ subspace of the compact space ${}^{\omega}(\omega_1 + 1)$ (with the appropriate product topology). To see this, observe that $Z = \bigcap_{i>0} O_i$ where $O_i = \bigcup_{\sigma \in Z} N_i(\sigma)$.

6.2. Z is not N-compact

The proof of this is similar to Nyikos' proof that Δ is not N-compact in [5]. Let $\mathscr{D} = \{D$: there is an $n \in \omega$ such that D is a full subset of ${}^nS\}$. One can show that $\{O$: O is clopen and O contains $\overline{\langle (0); D \rangle}$ for some $D \in \mathscr{D}\}$ forms a free clopen ultrafilter on Z with the countable intersection property.

Acknowledgement

The author wishes to express his appreciation to the referee for many thoughtful and valuable comments.

References

[1] W.G. Fleissner, The normal Moore space conjecture and large cardinals, in: J. Vaughan and K. Kunen, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 733–760.
[2] K. Kunen, Set Theory: An Introduction to Independence Proofs (North-Holland, Amsterdam, 1979).
[3] S. Mrowka, N-compactness, metrizability and covering dimension, in: Lecture Notes in Pure and Applied Mathematics 95 (Marcel Dekker, New York, 1985) 247-275.
[4] K. Nagami, Dimension Theory (Academic Press, New York, 1970).
[5] P.J. Nyikos, Prabir Roy's Δ is not N-compact, Gen. Topology Appl. 3 (1973) 197-210.
[6] A. Ostaszewski, A note on the Prabir Roy space, Preprint.
[7] P. Roy, Failure of equivalence of dimension concepts for metric spaces, Bull. Amer. Math. Soc. 68 (1962) 609–613.
[8] P. Roy, Nonequality of dimension for metric spaces, Trans. Amer. Math. Soc. 134 (1968) 117–132.